Algebraic Curves : Solutions sheet 1

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Unless otherwise specified, k is an algebraically closed field.

Exercise 1. Let $n \geq 1$ and $I, J \subseteq k[X_0, \dots, X_n]$ be ideals. For $d \geq 0$ we denote by $k[X_0, \dots, X_n]_d$ the subspace of forms of degree d and $I_d = I \cap k[X_0, \dots, X_n]_d$ (resp. $J_d = J \cap k[X_0, \dots, X_n]_d$). Show that:

1. If I, J are homogeneous, then $I + J$, IJ and $rad(I)$ are homogeneous.
2. If I is homogeneous, I is prime if, and only if, for all *homogeneous* $f, g \in k[X_0, \dots, X_n]$, $fg \in I \Rightarrow f \in I$ or $g \in I$.
3. I is homogeneous if, and only if, $I = \bigoplus_{d \geq 0} I_d$ (the right-hand side being a direct sum of abelian groups). Give an example of how this fails for non-homogeneous ideals.
4. If I is homogeneous, then there is a well-defined notion of forms of degree d in $\Gamma = k[X_0, \dots, X_n]/I$ and the corresponding spaces $\Gamma_d,\ d \geq 0$ are finite-dimensional over k.

Solution 1.

1. Assume I, J homogenous. Fix homogenous generators $f_1, \dots, f_n$ of I and $g_1, \dots, g_n$ of J. Then

$$I + J = \langle f_i, g_j \mid i, j \rangle$$

$$IJ = \langle f_i g_j \mid i, j \rangle$$

   are also homogenous, as products of homogenous polynomials are also homogenous. We can delay the proof for $\mathsf{Rad}(I)$ at question (3).

2. Clearly, I prime implies for all homogeneous $f, g \in k[X_0, \dots, X_n]$, $fg \in I \Rightarrow f \in I$ or $g \in I$.

   Now suppose that I is an homogenous polynomial, such that for all homogeneous $f, g \in k[X_0, \dots, X_n]$, $fg \in I \Rightarrow f \in I$ or $g \in I$. Let $f, g \in k[X_0, \dots, X_n]$ such that $fg \in I$. Take their decomposition in homogenous component $f = \sum_{i=0}^{k} f_i$ and $g = \sum_{i=0}^{k'} g_i$. Here f_i, g_i are homogenous of degree i. Now $f_k g_{k'}$ is homogenous of degree kk'. It is also the highest degree homogenous component of fg. As I is homogenous, (using (3)) $f_k g_{k'} \in I$. Now using the assumption, either f_k or g_k is in I. Suppose $f_k \notin I$. Then the degree $kk' - 1$ homogenous component of fg is $f_k g_{k'-1} + g_{k'} f_{k-1} \in I$. $g_k \in I$ implies that $f_k g_{k'-1} \in I$. By assumption, $g_{k'-1} \in I$. The proof finishes after k such iterations of the same argument.

3. This part is the key characterisation of homogenous polynomials. It means that I is homogenous if and only if, for all $f \in I$, homogenous components f_j of f are in I.

   It fails for non homogenous polynomial : take $J = \langle y - x^2 \rangle \subset k[x, y]$. Then y and x^2 are homogenous components of an element of J but they are not in J.

   Suppose that I is generated by a set of homogenous polynomial $\{h_i \mid i \in I\}$. Suppose $f \in I$. Take its decomposition of $f = \sum_j f_j$ into homogenous components. f also decomposes as $\sum_i a_i h_i$. As h_i's are homogenous, it is easy to see that $f_j = \sum_{i'} a_{i'} h_{i'}$ and thus $f_j \in I$.

   Suppose that I is an ideal, such that for all $f \in I$, if $f = \sum_j f_j$ is its decomposition in homogenous component, then for all j, $f_j \in I$.

   Now we can finish to prove that if I homogenous implies $Rad(I)$ homogenous. Let $f \in Rad(I)$, with homogenous decomposition $f = \sum_{j=0}^{d} f_j$. There is an $n \geq 0$ such that $f^n \in I$. Now the highest degree homogenous component of f^n is just f_d^n. So $f_d^n \in I$, so $f_d \in Rad(I)$. Now just conclude by induction on degree, using that nox $f - f_d \in Rad(I)$.

4. The decomposition $I = \oplus I_d$ respect the grading of $k[X_0, \ldots, X_n]$, meaning $I_d \subset k[X_0, \ldots, X_n]_d$. It is a general fact of modules (to check) that in this case quotient commutes with direct sum. $\Gamma_d = k[X_0, \ldots, X_n]_d / I_d$ is finite dimensional because $k[X_0, \ldots, X_n]_d$ is.

Exercise 2. Let $R = k[X, Y, Z]$ and $F \in R$ be an irreducible form of degree $n \geq 1$. Consider $V = V(F) \subseteq \mathbb{P}_k^2$ and $\Gamma = R/(F)$. For $d \geq 0$, we denote by Γ_d the subspace of forms of degree d in Γ (see previous exercise).

1. Construct an exact sequence $0 \to R \overset{\times F}{\to} R \to \Gamma \to 0$, where $\times F$ denotes multiplication by F in R.

2. Show that, for $d > n$:
$$dim_k(\Gamma_d) = dn - \frac{n(n-3)}{2}$$

Solution 2.

1. To show that $0 \to R \overset{\times F}{\to} R \to \Gamma \to 0$ is exact, we can say that :

   - $R \to R$, $f \mapsto f \cdot F$ defines a group morphism, injective since $F \neq 0$ and R is a domain. The image is (F).
   - (F) is the kernel of the quotient map $R \to \Gamma$. Quotient maps are always surjective.

2. We can refine the previous sequence with the grading : $0 \to R_{d-n} \overset{\times F}{\to} R_d \to \Gamma_d \to 0$. Now $dim_k(\Gamma_d) = dim_k(R_d) - dim_k(R_{d-n})$. The dimension of forms of degree d in $k[X_0, \ldots, X_N]$ is given by $\binom{d+N-1}{N-1}$.

   Indeed, a choice of an element of the basis is given by choosing the position of $N-1$ bars in $d+N-1$ locations (stars). For example, " $**|*|*$ " would represent the 4−form x^2yz.

   Now $\binom{d+2}{2} - \binom{d-n+2}{2}$ gives the desired expression.

Exercise 3. Let $V = V(Y - X^2, Z - X^3) \subseteq \mathbb{A}_k^3$. Show that:

1. $I(V) = (Y - X^2, Z - X^3)$.

2. $ZW - XY \in I(V)^* \subseteq k[X, Y, Z, W]$, but $ZW - XY \notin ((Y - X^2)^*, (Z - X^3)^*)$.

In particular, this shows that, for $F_1, \dots F_r \in k[X_1, \dots, X_n]$, the following inclusion can be strict: $(F_1^*, \dots, F_r^*) \subseteq (F_1, \dots, F_r)^*$.

Solution 3.

1. Set $I = (Y - X^2, Z - X^3)$. Since $k[X, Y, Z]/I \simeq k[X]$ is reduced I is radical, hence $I = I(V)$.

2. $Z - XY = Z - X^3 - X(Y - X^2) \in I$, so $ZW - XY \in I^*$.

   However, $(Y - X^2)^* = WY - X^2$, $(Z - X^3)^* = W^2 Z - X^3$. It follows that for $f \in ((Y - X^2)^*, (Z - X^3)^*)$, all monomial in Y of f are divisible by W, X^2 or X^3. Hence $ZW - XY \notin ((Y - X^2)^*, (Z - X^3)^*)$

Exercise 4. Let $n \geq 1$ and $T : \mathbb{A}_k^{n+1} \to \mathbb{A}_k^{n+1}$ be a linear change of coordinates (i.e. a linear automorphism of k^{n+1}). As it preserves lines through the origin it induces $T : \mathbb{P}_k^n \to \mathbb{P}_k^n$, what we call a *projective change of coordinates.*

1. Show, that one can send any $n+1$ points in $\mathbb{P}^n$ not lying on a hyperplane to any other $n+1$ points not lying on a hyperplane via a linear change of coordinates.

2. Formulate and prove a similar statement for hyperplanes instead of points.

Solution 4.

1. Let $S = \{P_1, \dots, P_{n+1}\}$ be a set of $n+1$ points not lying on a hyperplane. It allows us to lift S to a k-basis $\hat{S}$ of k^{n+1}. Indeed, if there is a linear relation $\hat{P}_{n+1} = \sum_i a_i \hat{P}_i$. Consider $H = Vect(P_1, \dots, P_n) \subset k^{n+1}$. H preserves lines so it induces a linear subspace $h \subset \mathbb{P}^n$, contained in a hyperplane. Then S is contained in h which contradicts the hypothesis. The same work for the target points, whose induced basis of k^{n+1} is denoted $\hat{T}$

   Now, it suffices to take the matrix $A \in \mathsf{GL}_n$ which sends $\hat{S}$ to $\hat{T}$. Using the short exact sequence

   $$0 \to k^* \xrightarrow{\lambda \mapsto \lambda I_n} GL_n \to PGL_n \to 0$$

   we get the desired linear linear change of coordinate $\bar{A} \in PGL_n$

2. There is a duality isomorphism $\mathbb{P}^n \simeq (\mathbb{P}^n)^*$, where $(\mathbb{P}^n)^*$ parametrises hyperplanes in $\mathbb{P}^n$.

   $$[y_0, \dots, y_n] \mapsto H \ : \ y_0 X_0 + \dots + y_n Y_n = 0$$

   The duality transform the condition "$n+1$ points not lying on a hyperplane" into "$n+1$ hyperplanes whose (global) intersection is empty" We can verify it as follows. Let $P = [x_0, \dots, x_n] \in \cap_{i=0}^n h_i$ with h_i hyperplanes, defined by the equation

   $$h_i \ : \ a_{i,0} X_0 + \dots + a_{i,n} X_n = 0$$

   Now, we get

   $$a_{0,0} x_0 + \dots + a_{0,n} x_n$$

   $$\dots$$

   $$a_{n,0} x_0 + \dots + a_{n,n} x_n$$

Then the $n+1$ points $A_i = [a_{i,0}, \dots, a_{i,n}]$ are all contained in the hyperplane

$$H \; : \; x_0 X_0 + \cdots + x_n X_n = 0$$

Exercise 5. Show that any two distinct lines in $\mathbb{P}^2_k$ intersect in one point.

Solution 5. Let L and L' be lines defined by the (homogenous) equation

$$ax + by + cz = 0$$

$$a'x + b'y + c'z = 0$$

Assume without lost of generality that $a \neq 0$ and $b' \neq -a^{-1}b$. Then, $x = a^{-1}(by + cz)$

$$y = \frac{-a^{-1}c + c'}{a^{-1}b + b'} z$$

and similarly

$$x = (\frac{-a^{-1}c + c'}{a^{-1}b + b'} + a^{-1}c) z$$

This defines a unique point in $\mathbb{P}^2_k$.

Exercise 6. Let $m, n \geq 1$ and $N = (n+1)(m+1) - 1 = mn + m + n$. We consider $\mathbb{P}^n_k$ with projective coordinates $X_0, \dots, X_n$, $\mathbb{P}^m_k$ with projective coordinates $Y_0, \dots, Y_m$ and $\mathbb{P}^N_k$ with projective coordinates $T_{00}, T_{01}, \dots, T_{0m}, T_{10}, \dots, T_{nm}$. We also denote the affine coverings of $\mathbb{P}^n_k$, $\mathbb{P}^m_k$, $\mathbb{P}^N_k$ associated to these coordinates as follows: $U_i = \{X_i \neq 0\}$, $V_j = \{Y_j \neq 0\}$ and $W_{ij} = \{T_{ij} \neq 0\}$.
Define the *Segre embedding* $S : \mathbb{P}^n_k \times \mathbb{P}^m_k \to \mathbb{P}^N_k$ by the formula:

$$S([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) = [x_0 y_0 : x_0 y_1 : \ldots : x_n y_m]$$

1. Show that S is well-defined and injective.
2. Let $Z = V(T_{ij}T_{kl} - T_{il}T_{kj},\ 0 \leq i, k \leq n,\ 0 \leq j, l \leq m) \subseteq \mathbb{P}^N_k$. Show that $S(\mathbb{P}^n_k \times \mathbb{P}^m_k) = Z$ (more specifically, $S(U_i \times V_j) = Z \cap W_{ij}$ for all i, j).
3. Show that the topology induced on $\mathbb{P}^n_k \times \mathbb{P}^m_k$ by the Zariski topology of $\mathbb{P}^N_k$ via the Segre embedding is different from the product topology.

Solution 6.

1. Well defined : the embedding is bilinear on each coefficient.

   Injective : Assume $S(x, y) = S(x', y')$. Without lost of generality, assume $x_0 = 1$. Take j such that $y_j \neq 0$. then $x_0 y_j = x'_0 y'_j \neq 0$. Then $x'_0 = \lambda \neq 0$. Then for all i, $y'_i = \lambda y_i$ so $y = y'$. Apply the same argument to y_j to get $x = x'$.

2. If $x_i \neq 0$, $y_j \neq 0$ then $S(x, y)_{ij} = x_i y_j \neq 0$. Hence $S(U_i \cap V_j) \subset W_{ij}$.

For the reverse inclusion, let $z = [z_{00}, z_{01}, \dots, z_{mn}]$ There is i, j such that $z_{ij} \neq 0$. WLOG assume $z_{ij} = 1$. Set for all i',

$$x_{i'} = z_{i'j}$$

and for all j',

$$y_{j'} = z_{ij'}$$

Using that

$$z_{ij} z_{i'j'} = z_{i'j} z_{ij'}$$

we get that $z_{i'j'} = x_{i'} y_{j'}$. This shows $W_{ij} \subset S(U_i \cap V_j)$.

3. We can mimic the familiar example of the diagonal in $\Delta \subset X \times X$, using the inclusion of one projective space in another, i.e. $\mathbb{P}^n \hookrightarrow \mathbb{P}^m$ with $n \leq m$. The image of this "diagonal" is closed in the Zariski topology of $\mathbb{P}_k^N$ but not in the product topology.